Grand canonical partition function

$$Q = \mathrm{Tr}\left(e^{-\beta\hat{H}+\beta\mu\hat{N}}\right) = \sum_{\{n_k\}} e^{\beta\mu\sum_k n_k - \beta\sum_k n_k \varepsilon_k}$$

Fermions: $n_k = 0,1$, $\eta = -1$

Bosons: $n_k \in \mathbb{Z}^+$, $\eta = +1$

$$\boxed{Q_\eta = \prod_k \left[1-\eta e^{\beta(\mu-\varepsilon_k)}\right]^{-\eta}}$$

Occupation statistics

$$\langle n_k\rangle = -\frac{1}{\beta}\partial_{\varepsilon_k}\ln Q_\eta = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-\eta}$$

Ideal gas

$\hat{H} = \frac{\hat{p}^2}{2m} \Rightarrow$ eigenstates are plane waves with wavevector $\vec{k}$

& eigenvalue $\varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m} = \varepsilon_k$

Degeneracy g

Spin $\sigma \in \{-S,\dots,S\} \Rightarrow$ there are $g = 2S+1$ $\neq$ eigenstates $|\vec{k},\sigma\rangle$ with energy $\varepsilon_{\vec{k}}$

Thermodynamics

Grand potential $G_\eta = -k_BT\ln Q_\eta = \eta k_BT\sum_{\vec{k},\sigma}\ln\left[1-\eta e^{\beta(\mu-\varepsilon_k)}\right]$

$$\boxed{G_\eta = \eta k_BTg\sum_{\vec{k}}\ln\left(1-\eta e^{\beta(\mu-\varepsilon_k)}\right)}$$

Pressure Definition $P = -\frac{\partial G_\eta}{\partial V}$. For an extensive system $G = -PV$, so that

$$\boxed{P = -\frac{\eta k_BT}{V}\sum_{\vec{k}}\ln\left[1-\eta e^{\beta(\mu-\varepsilon_k)}\right]}$$

Energy $\boxed{\langle E\rangle = \sum_{\vec{k},\sigma} \varepsilon_k \langle n_{\vec{k},\sigma}\rangle_\eta = g\sum_{\vec{k}} \frac{\varepsilon(\vec{k})}{e^{\beta(\varepsilon(\vec{k})-\mu)}-\eta}}$

Total particle number $\boxed{\langle N\rangle_\eta = \sum_{\vec{k}} \langle n_{\vec{k}}\rangle_\eta = g\sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)}-\eta} = -\partial_\mu \langle G\rangle}$

6.4) Black Body Radiation: the photon gas

Electromagnetic field without sources $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$

$\Rightarrow$ two polarizations $(\vec{\nabla}\cdot\vec{E}=0 \Leftrightarrow \vec{q}\cdot\vec{E}_q = 0)$

Energy $H = \frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0}$.

Expanding on normal modes leads to a sum of harmonic oscillators with frequencies $\omega(\vec{k},\alpha) = c|\vec{k}|$ $\Rightarrow$ Planck: treat as quantized oscillators

$\Rightarrow$ Same algebra as for the solid!

Energy & radiation energy

$$E = \sum_{\vec{k},\alpha}\left[\hbar\omega(\vec{k})\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar c|k|}-1}\right)\right] = E_0 + 2\sum_{\vec{k}} \frac{\hbar\omega}{e^{\beta\hbar c|k|}-1}$$

$\Rightarrow$ Energy of a boson gas with $g=2$. Indeed photons are massless bosons of spin 1, for which $\sigma=0$ is forbidden.

We also note that $\mu=0$, which is consistent with the fact that photon number is not conserved due to absorption & emission.

In a volume V, $\sum_{\vec{k}} = \frac{V}{(2\pi)^3}\int d^3\vec{k}$ leads to

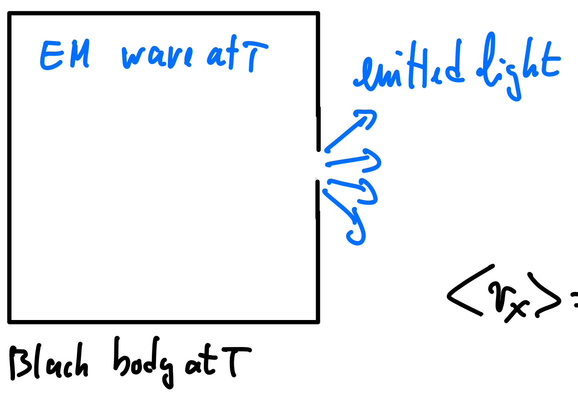
$$E = E_0 + V\int_0^\infty dk\, \underbrace{\frac{k^2}{\pi^2} k_B T \frac{\beta\hbar c k}{e^{\beta\hbar c k}-1}}_{e(k)\ \text{energy density of mode } k} = E_0 + V k_B T \frac{\pi^4}{15}\left(\frac{k_B T}{\hbar c}\right)^3$$

Radiation pressure

$$P = P_0 - \frac{k_B T}{\pi^2}\int_0^\infty dk\, k^2 \ln\left(1-e^{-\beta\hbar c k}\right)$$

$$= P_0 + \frac{k_B T}{3\pi^2}\int_0^\infty dk\, k^3 \frac{\beta\hbar c\, e^{-\beta\hbar c k}}{1-e^{-\beta\hbar c k}} = P_0 + \frac{E}{3V}$$

Black body radiation



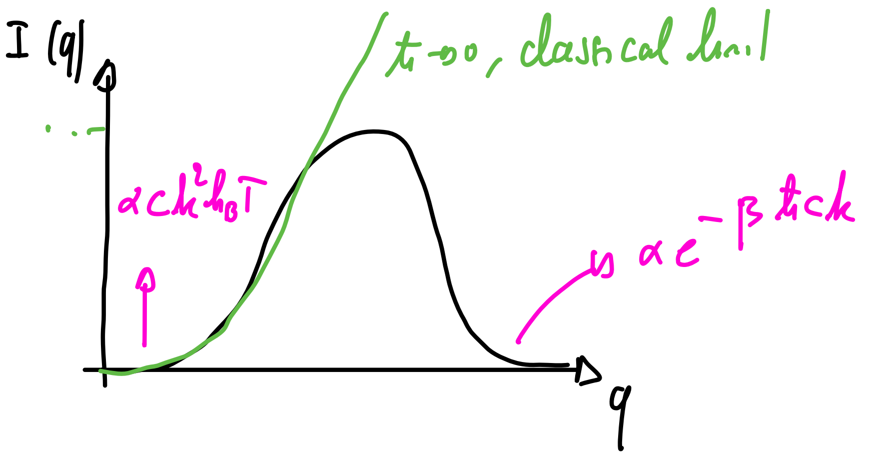
Flux of energy in k?

$$I(k) = \langle v_x\rangle\, e(k)$$

$$\langle v_x\rangle = \frac{1}{4\pi}\int_{\theta=0}^{\pi} d\theta \sin\theta \underbrace{\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{v_x>0} d\varphi\, \underbrace{c\sin\theta\cos\varphi}_{v_x} = \frac{c}{4\pi}\frac{\pi}{2}\,2 = \frac{c}{4}$$

$$\Rightarrow \boxed{I(k) = \frac{\hbar c^2}{4\pi^2}\frac{k^3}{e^{\beta\hbar c k}-1}}$$

Planck's law



$\Rightarrow$ the exponential cut-off suggested the quantization of light to Planck.

Total flux $= \frac{c}{4} \frac{E}{V} = \sigma T^4$, $\sigma = 5.67 \times 10^{-8}\ W.m^{-2}.K^{-1}$

Stefan's law

6.5) Bose-Einstein Condensation

6.5.1) Grand canonical ensemble

Ideal Bose gas of massive bosons with chemical potential μ

$$\langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \quad ; \quad \varepsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m} \Rightarrow \varepsilon_0 = 0$$

$\langle n_0 \rangle$ finite & positive $\Rightarrow -\infty < \mu < 0 \Leftrightarrow 0 < z = e^{\beta\mu} < 1$

Classical limit $\langle N \rangle = \frac{V}{\Lambda^3} z = \frac{V}{\Lambda^3} e^{\beta\mu} \Rightarrow \mu_{cl} = k_B T \ln\left[\langle \frac{N}{V} \rangle \Lambda^3\right]$

distance between particles $\sim \left(\frac{V}{\langle N \rangle}\right)^{1/3} \gg \Lambda \Rightarrow \boxed{\mu \to -\infty \ \& \ z \to 0}$

(Also reached through $k_B T \to \infty$)

What about $\mu \to 0$ & $z \to 1$? $\langle n_0 \rangle = \frac{1}{e^{-\beta\mu} - 1} \xrightarrow[\mu \to 0^-]{} \infty$

$$= \frac{z}{1-z} \xrightarrow[z \to 1^-]{} \infty$$

problem!

Occupation of energy levels

$$\rho_0 = \frac{N}{V} = \frac{g}{V} \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} = \underbrace{\frac{g}{V} \frac{1}{e^{-\beta\mu} - 1}}_{\rho_{gs} \text{ in ground state}} + \underbrace{\frac{g}{V} \sum_{\vec{k} \neq 0} \frac{1}{e^{\beta[\varepsilon(\vec{k}) - \mu]} - 1}}_{\rho_{es} \text{ in excited states}}$$

Excited states

$$\rho_{ES} = \frac{g}{(2\pi)^3}\int d^3\vec{k}\,\frac{1}{z^{-1}e^{\beta\frac{\hbar^2k^2}{2m}}-1} = \frac{g}{2\pi^2}\int_0^\infty dk\,\frac{k^2}{z^{-1}e^{\beta\frac{\hbar^2k^2}{2m}}-1}$$

$x = \frac{\hbar^2k^2}{2mk_BT}$

$k = \frac{\sqrt{2mk_BT}}{\hbar}\sqrt{x}$

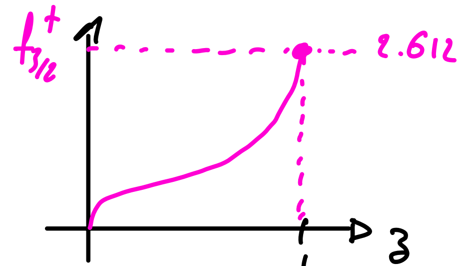
$dk = \frac{\sqrt{2mk_BT}}{\hbar}\frac{dx}{2\sqrt{x}}$

$$= \frac{g}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{x^{1/2}}{z^{-1}e^x-1}\,dx$$

$$\rho_{ES} = g\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2}\frac{\pi}{\sqrt{2}}\int_0^\infty dx\,\frac{x^{1/2}}{z^{-1}e^x-1}$$

$$\boxed{\rho_{ES}\lambda^3 = g\, f^+_{3/2}(z) \quad \text{where} \quad f^\eta_m(z) = \frac{1}{(m-1)!}\int_0^\infty dx\,\frac{x^{m-1}}{z^{-1}e^x-\eta}}$$

Comment: $f^+_{3/2}(1) = \int_0^\infty dx\,\frac{x^{1/2}}{e^x-1} = 2.612$ finite



Mathematically, $\frac{x^{1/2}}{e^x-1} \sim x^{-1/2}$ which is integrable at 0 since

$$\int_\varepsilon dx\, x^{-1/2} = 2\varepsilon^{1/2} \xrightarrow[\varepsilon\to 0]{} 0$$

For $z<1$, $\rho_{ES}(z) < \frac{g}{\lambda^3} f^+_{3/2}(1) \equiv \rho_{ES}^{MAX} = 2.612\, g\, \lambda^3$

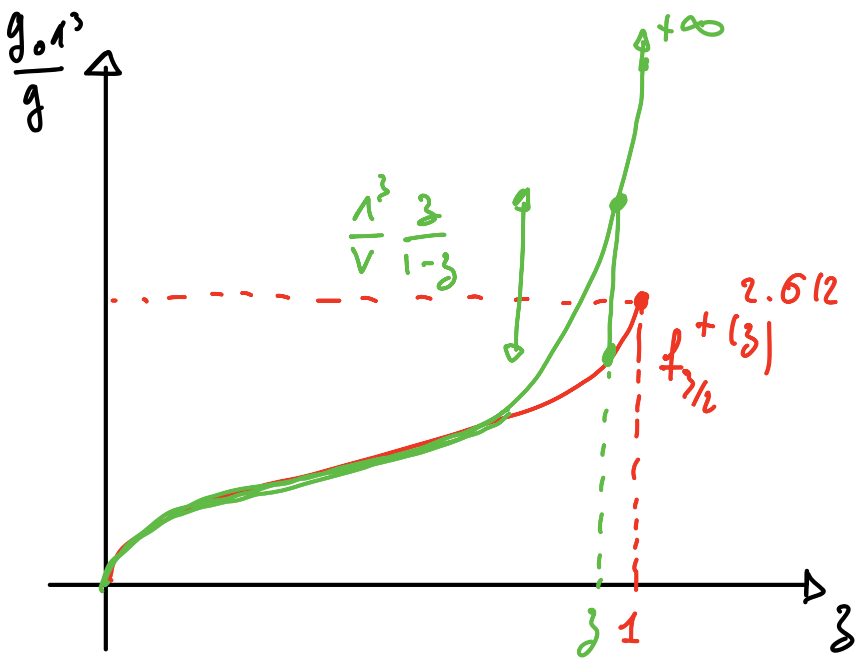
$$\boxed{\rho_0(z) = \frac{g}{V}\frac{z}{1-z} + \frac{g}{\lambda^3} f^+_{3/2}(z)}$$

As $z\to 1$ & $\mu\to 0$, ρ_{ES} remains finite but ρ_{GS} diverges. At fixed V, the majority of atoms end up in the ground state!

<u>Caveat</u>: Q finite requires $z<1$, as $V\to\infty$ $\frac{g}{V}\frac{z}{1-z}\to 0$

$\Rightarrow$ In the thermodynamic limit, no phase transition in the grand canonical ensemble.

The $z\to 1$ & $V\to\infty$ limit



Take $V \gg \Lambda^3$, for $z\sim\frac{1}{2}$, $\rho_{GS} \ll \rho_{ES}$

As $z\to 1$, $\rho_{GS}\propto \frac{1}{V}\frac{z}{1-z}\to\infty$

but as $V\to\infty$, $\rho_{GS}\to 0$ and

$$\boxed{\rho_0(z) \underset{V\to\infty}{\simeq} \rho_{ES} \Rightarrow \rho_0(z) \underset{V\to\infty}{\leq} \frac{g}{\Lambda^3}2.612}$$

Boundary layer: $z = 1-\frac{g}{\alpha V\rho_0} \Rightarrow \rho_{GS} = \frac{g}{V}\frac{1-\frac{g}{\alpha V\rho_0}}{\frac{g}{\alpha V\rho_0}} = \alpha\rho_0 - \frac{g}{V} \underset{V\to\infty}{\to} \alpha\rho_0$

For $\alpha\sim O(1)$, there is a finite fraction of bosons in the ground state $\Rightarrow$ Bose-Einstein condensation. Except that this boundary layer is not accessible experimentally g-c

6.5.2) Canonical ensemble

For large systems, the descriptions of intensive quantities like ρ_0

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& μ are expected to be equivalent in all ensembles. Let's thus switch to the canonical ensemble where we can fix $\rho_0 = \frac{N}{V}$.

Fixing density $\quad \rho_0 = \rho_{GS}(z) + \rho_{ES}(z)$ is now an equation for z.

If $\rho_0 < \rho_{ES}^{MAX}$, $z<1$, $\rho_0 = \underbrace{\rho_{GS}(z)}_{\substack{\to 0 \\ V\to\infty}} + \rho_{ES}(z) \underset{V\to\infty}{\simeq} \rho_{ES}(z)$

If $\rho_0 > \rho_{ES}^{MAX}$, this is impossible, $z \to 1$ & $\rho_{GS} = \rho_0 - \rho_{ES}^{MAX}$ finite

$\Rightarrow$ True Bose Einstein Condensation with a finite fraction $\alpha = \frac{\rho_{GS}}{\rho_0}$ in the ground state.

Then, $\langle n_0 \rangle = \alpha V \rho_0 = \frac{g}{z^{-1}-1} \Rightarrow z^{-1} = 1 + \frac{g}{\alpha V \rho_0}$ & $z \simeq 1 - \frac{g}{\alpha V \rho_0}$ as before

Is n_0 the sole macroscopically occupied state?

$$\langle n_1 \rangle = \frac{1}{z^{-1} e^{\beta \varepsilon_1} - 1} \quad ; \quad k_1 = \frac{2\pi}{L} \Rightarrow \beta \varepsilon_1 = \beta \frac{\hbar^2}{2m} \frac{4\pi^2}{L^2} \equiv \frac{K}{L^2}$$

$$\langle n_1 \rangle = \left[\left(1 + \frac{g}{\alpha L^3 \rho_0}\right)\left(1 + \frac{K}{L^2}\right) - 1 \right]^{-1} \simeq \left[\frac{K}{L^2} + \frac{1}{\alpha \rho_0 L^3} \right]^{-1} \sim \frac{L^2}{K}$$

$$\rho_1 = \frac{\langle n_1 \rangle}{V} \sim \frac{1}{L} \underset{V\to\infty}{\longrightarrow} 0$$

Only the ground state has a macroscopic number of particle

Transition temperature

μ or N are not the easiest control parameter, but we can tune $\rho_{ES}^{MAX} = g\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} f^+_{3/2}(1)$ by changing T.

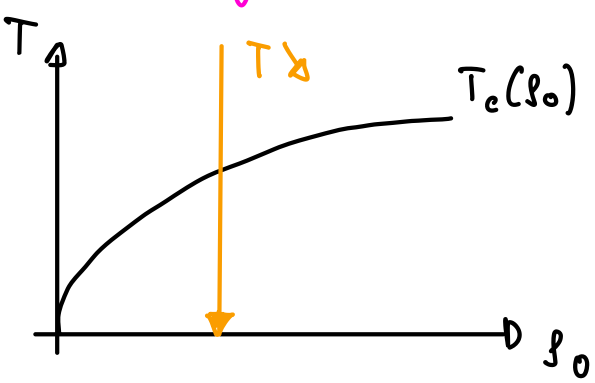
* $T > T_c$, $\rho_{ES}^{MAX} > \rho_0$, $z < 1$ & $\rho_{GS} \xrightarrow[V\to\infty]{} 0$ $\Rightarrow$ No BEC

* $T = T_c$, $\rho_0 = \rho_{ES}^{MAX}$ $\Rightarrow$ $\boxed{k_B T_c = \frac{h^2}{2\pi m}\left(\frac{\rho_0}{g\, f^+_{3/2}(1)}\right)^{2/3}}$

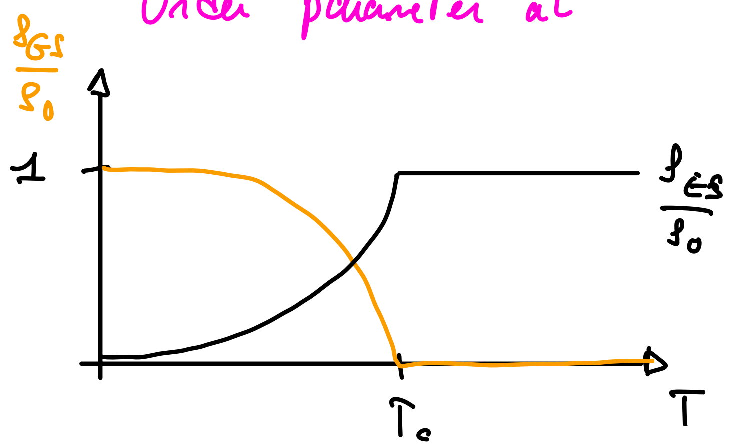
* $T < T_c$, $\rho_{GS} = \rho_0 - \underbrace{\rho_{ES}^{MAX}(T)}_{\propto T^{3/2}}$. Since $\rho_0 = \rho_{ES}^{MAX}(T_c)$, we have

$$\boxed{\frac{\rho_{GS}}{\rho_0} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$

Phase diagram



Order parameter at



Comment: Why can we use $\rho_0 = \rho_{GS}(z) + \rho_{ES}(z)$ in the canonical ensemble while we derived it in the grand canonical ensemble?

[Crisanti, Salasnich, Sarracino, Zannetti, Arxiv: 2404.17300]

Thermodynamics

Grand potential: Treating the GS separately

$$G = k_B T g \left[\ln(1-z) + \frac{V}{(2\pi)^3} \int dk\, 4\pi k^2 \ln\left(1 - z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right) \right]$$

$$x = \frac{\hbar^2 k^2}{2 m k_B T} \Rightarrow k = \sqrt{x} \sqrt{\frac{8\pi^2 m k_B T}{h^2}}$$

$$G = k_B T g \ln(1-z) + \frac{g V k_B T}{4\pi^2} \left(\frac{8\pi^2 m k_B T}{h^2} \right)^{3/2} \underbrace{\int dx\, x^{1/2} \ln(1 - z e^{-x})}_{\text{IBP } -\frac{2}{3} \int dx \frac{x^{3/2} z e^{-x}}{1 - z e^{-x}}}$$

$$G = k_B T g \ln(1-z) - \frac{g V k_B T}{\Lambda^3} \underbrace{\underbrace{\frac{2}{3} \frac{2}{\sqrt{\pi}}}_{\frac{1}{\frac{3}{2}!}} \int dx \frac{x^{3/2}}{z^{-1} e^{x} - 1}}_{f^+_{5/2}(z)}$$

$$\boxed{G = k_B T g \ln(1-z) - \frac{g V k_B T}{\Lambda^3} f^+_{5/2}(z)}$$

Pressure: $P = -\frac{\partial G}{\partial V} = \frac{g k_B T}{\Lambda^3} f^+_{5/2}(z)$ $\Rightarrow$ the GS bosons do not contribute to the pressure.

This makes sense: $\vec{k}_0 = 0$ so that $\vec{p}_0 = \hbar \vec{k}_0 = 0 \Rightarrow$ no momentum to transfer

$T < T_c$ $\quad P = \frac{g k_B T}{\Lambda^3} f^+_{5/2}(1) \simeq 1.31 \frac{g k_B T}{\Lambda^3} \Rightarrow$ independent from N & V!

$T > T_c$ $\quad \rho_0 \simeq \frac{g}{\Lambda^3} f^+_{3/2}(z) \Rightarrow P = \rho_0 k_B T \frac{f^+_{5/2}(z)}{f^+_{3/2}(z)}$

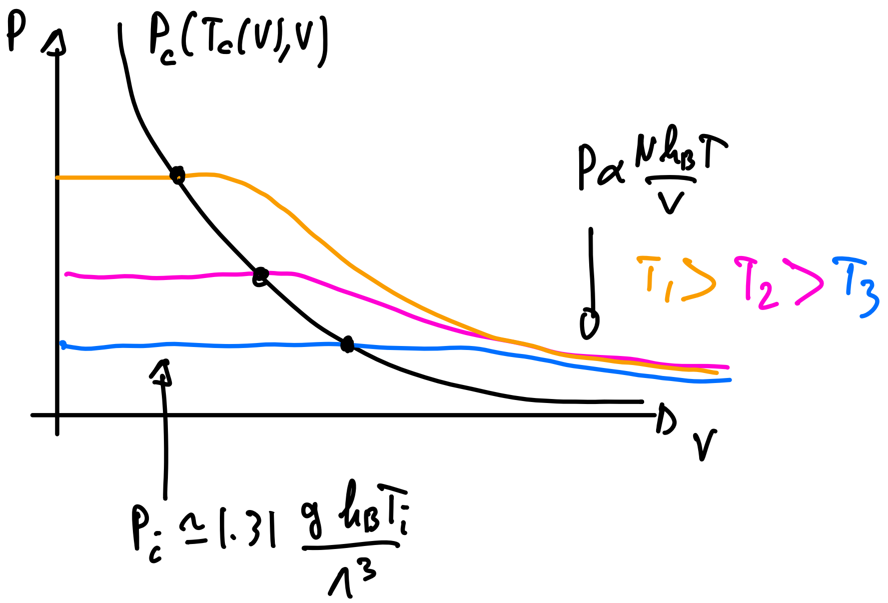
$T \gg T_c$; $z \ll 1$, $f_m^+(z) = \frac{1}{(m-1)!}\int_0^\infty dx \, \frac{x^{m-1}}{z^{-1}e^{x}-1} \simeq \frac{z}{(m-1)!}\underbrace{\int_0^\infty dx\, x^{m-1}e^{-x}}_{(m-1)!}$

$\Rightarrow P \xrightarrow[T \gg T_c]{} \rho_0 k_B T$ as expected.

$T = T_c$ $\quad \rho_0 = \frac{N}{V} = \frac{g f^+_{3/2}(1)}{h^3}(2\pi m k_B T_c)^{3/2} \Rightarrow T_c(V) = \frac{1}{2\pi m k_B T_c}\left(\frac{N h^3}{V g f^+_{3/2}(1)}\right)^{2/3}$

$\Rightarrow$ At T_c, $P_c(T_c(V), V) \propto \frac{N T_c(V)}{V} \sim \frac{1}{V^{5/3}}$

Isotherm $P(V)$



High temperature expansion: how to connect to classical stat mech?

$f_m^+(z) \simeq z \Rightarrow$ leading order term $\Rightarrow$ what about higher orders?

$$f_m^+(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m} \simeq z + \frac{z^2}{2^m} + \frac{z^3}{3^m} + \dots$$

$u = x(1+k)$

$$f_m^+(z) = \frac{1}{(m-1)!}\int_0^\infty dx\, x^{m-1} z e^{-x} \sum_{k=0}^{\infty} (z e^{-x})^k = \frac{1}{(m-1)!}\sum_{k=0}^{\infty} z^{k+1} \underbrace{\int_0^\infty dx\, x^{m-1} e^{-x(1+k)}}_{\frac{(m-1)!}{(1+k)^m}}$$

From here $\Rightarrow$ P as a series in z and ρ_0 as a series in z $\Big\}$ $\Rightarrow P$ as a series in n.

Energy & heat capacity:

$$\langle E\rangle = \partial_\beta(\beta G) = \frac{3gV}{\Lambda^4} f^+_{5/2}(z)\frac{\partial\Lambda}{\partial\beta} \quad ; \Lambda = \sqrt{\frac{h^2\beta}{2\pi m}} \Rightarrow \partial_\beta \Lambda = \frac{k_BT}{2}\Lambda$$

$$\langle E\rangle = \frac{3}{2}k_BT\,\frac{gV}{\Lambda^3} f^+_{5/2}(z) = \frac{3}{2}PV$$

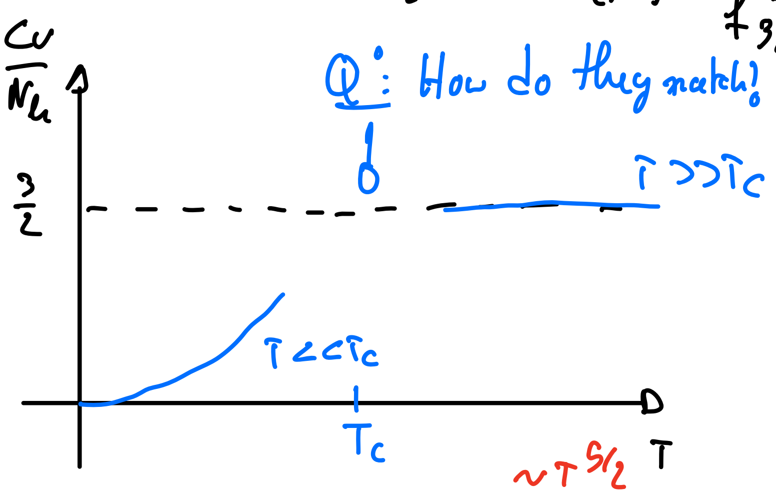
Low temperature limit

$T > T_c$, $\rho_{GS}=0$ & $N \underset{V\to\infty}{\simeq} \frac{gV}{\Lambda^3} f^+_{3/2}(z) \Rightarrow E = \frac{3}{2}Nk_BT\,\frac{f^+_{5/2}(z)}{f^+_{3/2}(z)}$

$T \gg T_c$ leads to $E \simeq \frac{3}{2}Nk_BT$ & $C_V = \frac{3}{2}N$.

$T = T_c$, $\rho_0 = \frac{g}{\Lambda_c^3} f^+_{3/2}(1)$ & $gV = \frac{N\Lambda_c^3}{f^+_{3/2}(1)}$

$T < T_c$, $\langle E\rangle = \frac{3}{2}Nk_BT\left(\frac{\Lambda_c}{\Lambda}\right)^3 \frac{f^+_{5/2}(1)}{f^+_{3/2}(1)} \propto NT^{5/2} \Rightarrow C_V \propto T^{3/2}N$



$T > T_c$; $\langle E\rangle = \frac{3}{2}\underbrace{k_BT\,\frac{gV}{\Lambda^3}}_{\sim T^{5/2}} f^+_{5/2}(z)$

$$C_V = \frac{3}{2} k_B T \frac{gV}{\lambda^3}\left[\frac{5}{2T} f^+_{5/2}(z) + \underbrace{\frac{\partial z}{\partial T}}_{②} \cdot \underbrace{\frac{\partial}{\partial z} f^+_{5/2}(z)}_{①}\right]$$

① Direct algebra $\partial_z f^+_m = \frac{1}{z} f^+_{m-1}$

$$\frac{\partial}{\partial z}\int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x-1} = -\int_0^\infty dx \frac{x^{m-1}}{(z^{-1}e^x-1)^2}\left(-\frac{1}{z^2}e^x\right) = \frac{1}{z}\int_0^\infty dx \frac{\overbrace{z^{-1}e^x}^{= \frac{\partial}{\partial x}\left[-\frac{1}{z^{-1}e^x-1}\right]}}{(z^{-1}e^x-1)^2} x^{m-1}$$

$$\overset{IBP}{=} \frac{m-1}{z}\int_0^\infty dx \frac{x^{m-2}}{z^{-1}e^x-1}$$

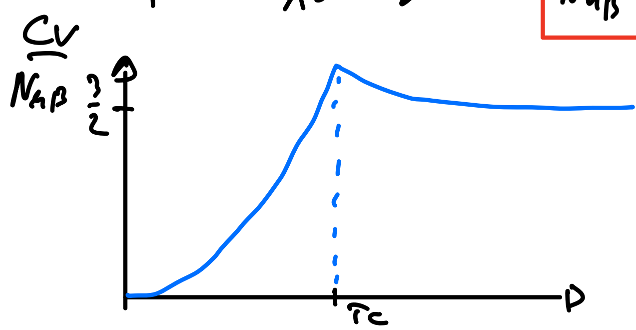
Multiplying both sides by $\frac{1}{m!}$ leads to $\partial_z f^+_m = \frac{1}{z} f^+_{m-1}$

② $\rho_0\lambda^3 = g f^+_{3/2}(z) \Rightarrow \frac{\partial}{\partial T}\ln(\rho_0\lambda^3) = -\frac{3}{2T} = \frac{\partial_T f^+_{3/2}(z)}{f^+_{3/2}(z)} = \frac{1}{z}\frac{f^+_{1/2}(z)}{f^+_{3/2}(z)}\frac{\partial z}{\partial T}$

$$\Rightarrow \frac{\partial z}{\partial T} = -\frac{3z}{2T}\frac{f^+_{1/2}(z)}{f^+_{3/2}(z)}$$

① + ② $\Rightarrow$ $$\boxed{C_V = \frac{15}{4} k_B \frac{gV}{\lambda^3} f^+_{5/2}(z) - \frac{9}{4} k_B \frac{gV}{\lambda^3}\frac{f^+_{3/2}(z)^2}{f^+_{1/2}(z)}}$$

As $T \to T_c$, $\left.\begin{array}{l} f^+_{5/2} \to 1.34 \\ f^+_{3/2} \to 2.61 \\ f^+_{1/2} \to \infty \end{array}\right\}$ $C_V \to \frac{15}{4} k_B \overbrace{\frac{gV}{\lambda^3}}^{N/f^+_{3/2}(1)} f^+_{5/2}(1)$ & $\boxed{\frac{C_V}{Nk_B} \simeq 1.92 > \frac{3}{2}}$



High temperature expansions The approach to $T = T_c$ ($T > T_c$) relies on a high-T expansion, which leads to $\frac{C_V}{N k_B} \simeq \frac{3}{2}\left(1 + \rho_0 \frac{\lambda^3}{2^{7/2}} + \dots\right) \geq \frac{3}{2}$

6.6) Fermi-Dirac Statistics

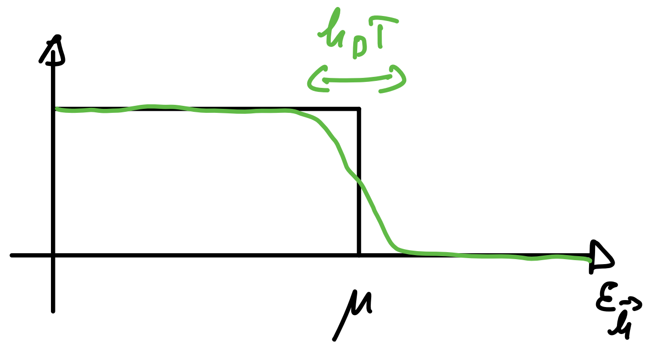
6.6.1) The $T \to 0$ limit

$$\langle n_{\vec{k},\sigma} \rangle = \frac{1}{e^{\beta[\varepsilon_{\vec{k}} - \mu]} + 1}$$

$$\frac{1}{1+e^{x}} \begin{cases} \to 0 & x \to -\infty \\ \to +\infty & x \to +\infty \end{cases}$$

$$\frac{1}{1+e^{x}} = 1 - \frac{1}{1+e^{-x}} \Rightarrow \text{symmetric with respect to } (0, 1/2)$$

Occupation statistics at low T



At $T=0$, all energy levels are full up to the Fermi energy $\varepsilon_F = \mu$.

The occupied levels are called the Fermi sea.

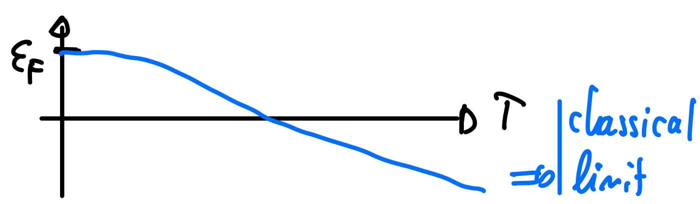
They satisfy $\varepsilon_k < \varepsilon_F \Leftrightarrow \frac{\hbar^2 k^2}{2m} < \varepsilon_F = \mu$ & $k_F = \sqrt{\frac{2m\mu}{\hbar^2}}$ is Fermi wavenumber.

Density $N = \sum_{|\vec{k}| < k_F} g = g \frac{V}{(2\pi)^3} \int_{|k| < k_F} d^3\vec{k} \Rightarrow \boxed{N = g \frac{V}{6\pi^2} k_F^3}$

Conversely $\boxed{k_F = \left(\frac{6\pi^2 \rho_0}{g}\right)^{1/3}}$ & $\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g}\right)^{2/3}} \equiv k_B T_F$

Fermi temperature

Canonical perspective: Fix $N, V, T \Rightarrow \mu(T)$



We expect that an energy $k_B T$ allows a single particle to reach energy levels up to $\varepsilon_n \simeq k_B T$.

If $k_B T \ll \varepsilon_F$, the system is close to its zero temperature limit. If $k_B T \gg \varepsilon_F$, thermal fluctuations are expected to be very important.

The example of metals : consider a cristal of atoms + electrons at room temperature, $T = 300°K$.

E.g. Copper
$$\left.\begin{array}{l} \rho_m = \frac{M}{V} = 9\,g/cm^3 \\ M = m\mathcal{N}_a = 63.5\,g/mol \end{array}\right\} \rho_0 = \frac{\rho_m}{m} = \mathcal{N}_a \frac{\rho_m}{M} \simeq 10^{29} m^{-3}$$

$\Rightarrow$ distance between atoms $d = \frac{1}{(\rho_0)^{1/3}} \simeq 10^{-10} m$

$$\lambda_{Cu} = \sqrt{\frac{h^2}{2\pi m k_B T}} \simeq 1.3 \times 10^{-11} m << d \Rightarrow \text{atoms} \sim \text{classical}$$

$$\lambda_{e^-} = \sqrt{\frac{h^2}{2\pi m_e k_B T}} = 40 \times 10^{-10} m \gg d \Rightarrow \text{important quantum effect.}$$

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2\rho_0}{g}\right)^{2/3} = 7eV \quad \text{vs} \quad k_B T = 0{,}024\ eV$$

$T_F = 10^4 K \Rightarrow$ the e^- form a Fermi fluid at very low temperature

Q: What are the thermodynamic properties of fermions at small but finite temperature.

6.6.2) Thermodynamics at low temperatures

$$G = -g k_B T \sum_{\vec{k}} \ln\left[1 + z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right] \simeq -g k_B T \frac{V}{2\pi^2} \int dk\, k^2 \ln\left(1 + z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right)$$

$x = \frac{\hbar^2 k^2}{2m k_B T}$; $k = \sqrt{x} \sqrt{\frac{2\pi m k_B T}{h^2}}\, 2\sqrt{\pi}$; $dk = \frac{dx}{\sqrt{x}} \frac{\sqrt{\pi}}{\lambda}$

$$G = -g k_B T \frac{V}{\lambda^3} \frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty dx\, x^{1/2} \ln(1 + z e^{-x})}_{\frac{2}{3} \frac{x^{3/2} z e^{-x}}{1 + z e^{-x}}} \Rightarrow \boxed{G = -g k_B T \frac{V}{\lambda^3} f^-_{5/2}(z)}$$

G is extensive $\Rightarrow$ $\boxed{P = -\frac{G}{V} = g \frac{k_B T}{\lambda^3} f^-_{5/2}(z)}$

$$\langle N \rangle = z\, \partial_z \ln Q = -\beta z\, \partial_z G = g \frac{V}{\lambda^3} f^-_{3/2}(z) \quad \text{since } z\,\partial_z f^-_m(z) = f^-_{m-1}(z)$$

$$\Rightarrow \boxed{\rho_0 = \frac{g}{\lambda^3} f^-_{3/2}(z)}$$

$$\boxed{\langle E \rangle = \partial_\beta (\beta G)_z = \frac{3}{2} k_B T \frac{gV}{\lambda^3} f^-_{5/2}(z) = \frac{3}{2} PV}$$ (like for Bosons)

Low temperature expansion

We know that $\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} \rho_0 = f^-_{3/2}(z) = \frac{\sqrt{\pi}}{2} \int_0^\infty dx \frac{z x^{\frac{1}{2}}}{e^{x} + z}$

As $T \to 0$, $f^{-}_{3/2}(z) \to \infty \Rightarrow$ requires z large.

This is consistent with $\beta \to \infty$ & $\mu \to \varepsilon_F > 0$ so that $z = e^{\beta\mu} \to +\infty$.